The Coefficient of Restitution of Rebounding Capsules

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ABSTRACT

This research has been done to solve one of the 34th IYPT problems . The question statement is as follows: "A spherical ball dropped onto a hard surface will never rebound to the released height, even if it has an initial spin. A capsule-shaped object (i.e. Tic Tac mint) on the other hand may exceed the initial height". To find the relevant parameters qualitative analysis by *Newtonian* and *Lagrangian* mechanics has been done.

Key words: *Rebounding Capsule, Newtonian Mechanics, Lagrangian Mechanics*

1 Introduction

When tablets collide during manufacturing and handling operations they rebound with a force and velocity that is determined by the collision conditions and the properties of the materials. This collision rebound behavior of solid bodies can be described using a parameter known as the "coefficient of restitution" (CoR). The CoR varies with the mechanical properties of both colliding bodies and is lower for more plastic collisions and higher for elastic collisions.

2 Basic Theories

The coefficient of restitution, e, is a parameter that quantifies the energy losses during collision and is defined as (Eq. 1):

$$e = \frac{-V_r}{V_i} \quad (1)$$

where Vr and Vi are the rebound and impact relative velocities of the colliding objects. The collision is said to be perfectly elastic if e = 1 and completely inelastic if e = 0. For a freely falling object under the influence of gravity, Eq. (1) simplifies to the form [1](Eq. 2):

$$e = \sqrt{\frac{h_2}{h_1}} \quad (2)$$

To solve this problem the capsule-shaped object is divided by 2 hemispheres, then integrating them in each axis $(\dot{x}, \dot{y}, \dot{z})$ and summing up would be the main approach (Fig.1).

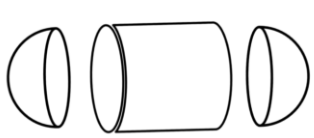


Fig. 1: capsule-shaped object is divided into 2 hemispheres

Defining force as a function of momentum-time, gives (Eq. 3):

$$F = \frac{dP}{dt}, \frac{mv_0'^2}{2} = mgh_0 \rightarrow v' = \sqrt{2gh_0}; \text{ then} \quad (3)$$

$$m\Delta v = N\Delta t = m(v_1' - v_0')$$

and for calculating N^th height of the capsule, it is released with: $v_{initial} = 0$ then :

$$E_1 = mgh_1 = \alpha mgh_0 \rightarrow h_1 = \alpha h_0 \mid h_2 = \alpha^2 h_0 \mid h_n = \alpha^n h_0 \quad (4)$$

where α is $0 < \alpha < 1$.

If we assume the second restitution as an ideal parabola (neglecting air drag), Then for the second height being greater than the initial height, $v^2 \sin^2\theta > h_{initial}$. $\sin^2\theta$ in $\frac{\pi}{2}$ will be maximized; but $v = \omega . r$ and if $\theta = \frac{\pi}{2}$ then the angle (ϕ), between the radius (r), the geometric center and impact ground and $(r \sin\phi)$ would be equal to 0; so:

$$\tau = rN \sin\phi = 0$$

Since there is waste force acting upon the capsule then it wouldn't reach the initial height. In this case:

$$v = -gt + v_0 \qquad v_0 = e\sqrt{2gh} \quad (5)$$

where e is the coefficient of restitution $\left(e = \frac{v_{rebound}}{v_{impact}}\right)$. So, the ideal angle (ϕ, θ) would be π/4.

Now three different shapes are analyzed:

a) Cylinder

$$I_z = \frac{mr^2}{2} \qquad I_y = I_x = \frac{1}{12}m(3r^2 + h^2) \quad (6)$$

b) Capsule

$$I_x = I_y = \frac{1}{12}m_1(3r_1^2 + h_1^2) \quad (7)$$

$$I_z = \frac{1}{2}m_1 r_1^2 + \frac{2}{10}m_2 r_2^2 + \frac{2}{10}mr^2 \quad (8)$$

c) Sphere

$$I_x = I_y = I_z = \frac{2}{10}mr^2 \quad (9)$$

Lagrangian Mechanics (Released Height – Impact Ground) :

$$L = T - V \qquad \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + mgx \quad (10)$$

Acquiring the equitation of motion by the Euler-Lagrange method:

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}}\right) - \left(\frac{\delta L}{\delta x}\right) = 0 \quad (11)$$

$$m\frac{d^2x}{dt^2} - kx + mg = 0 \quad (12)$$

In our 3D model the capsule is made out of two hemispheres and one cylinder where ($\xi = x^{\frac{3}{2}}$) [2], then we write our impact ΣF as followings:

$$m\frac{d^2y}{dt^2} = \left(-k\Delta y^{\frac{3}{2}} - \delta\left(\Delta\dot{y}^{\frac{3}{2}}\right)\right) \quad (13)$$

$$m\frac{d^2x}{dt^2} = \left(-x\Delta y^{\frac{3}{2}} - \delta\left(\Delta\dot{x}^{\frac{3}{2}}\right)\right) \quad (14)$$

$$\frac{d^2\theta}{dt^2} = -N\frac{\theta}{|\theta|}\frac{l}{2}\cos(\phi) \quad (15)$$

$$\Delta y = \left(y - \frac{\theta}{|\theta|}\frac{l}{2}\sin(\phi) - r\right) \quad (16)$$

$$\Delta x = (x - \frac{\theta}{|\theta|}\frac{l}{2}\cos(\phi) - x_{point}) \quad (17)$$

3 Experiments

For finding the ideal height for $h_2 > h_1 \mid h_4 > h_3 \mid h_3 > h_1$ we would write the PDF (Probability Distribution Function). When the capsule hit the ground it starts gong up and down but with two different motions linear and rotational. We need to know why this phenomenon happens (Figs. 2 and 3).

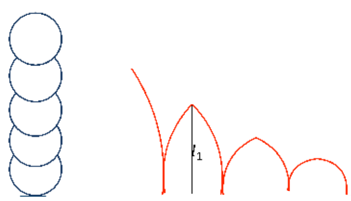


Fig.2: Capsule hits the ground without considering friction

$l_1 = \alpha mgh_0$

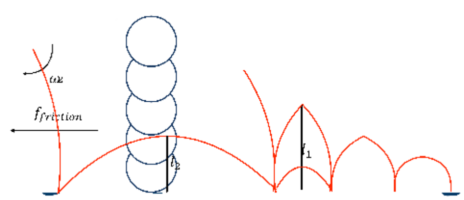


Fig.3: Capsule hits the ground with considering friction

$l_2 = \alpha mgh_0$

$l_1 > l_2$

In linear and rotional motion , total energy will be:

$$\Sigma E = \frac{mv^2}{2} + \frac{I\omega^2}{2} + mgh \quad (18)$$

$$mgh = \alpha mgh = mgh_1 + \frac{I\omega^2}{2} \quad (19)$$

$$\alpha mgh_0 = mgh_1 + \frac{I\omega^2}{2}$$

$$\alpha(mgh_0 + \frac{I\omega^2}{2}) = mgh_1 + \frac{I\omega^2}{2} \quad (20)$$

$$mgh + \frac{I\omega^2}{2} = \alpha mgh_1 + \alpha\frac{I\omega^2}{2} \quad (21)$$

$h_0 = dropheight$

$h_1 = reboundheighth_0$

$h_2 = reboundheighth_1$

$$h_2 = \alpha h_1 + \frac{I}{2mg}(\alpha\omega^2 - \alpha_2^2) \quad (22)$$

$$\alpha\omega_1^2 - \omega_2^2 > \frac{2mgh(1-\alpha)}{I} \quad (23)$$

$$\omega_1 > \sqrt{\frac{\omega_2^2}{\alpha} + \frac{2mgh_1}{I} + \frac{1-\alpha}{\alpha}} \quad (24)$$

When the capsule collide the ground the coefficient restitution which is a function of the velocity before and after hitting will be as follows:

$$e = \sqrt{\frac{KE_{after}}{KE_{before}}} = \sqrt{\frac{\frac{1}{2}mv^2}{\frac{1}{2}mu^2}} = \frac{v}{u} \quad (25)$$

$$PotentialEnergy \simeq 0$$

$$v^2 = u^2 + 2gh \quad (26)$$

$$a = \frac{v+u}{CollisionTime} \quad (27)$$

By Euler - Lagrangian equation:

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 - mg(x) \quad (28)$$

$$\frac{\delta L}{\delta x} - \frac{d}{dt}(\frac{\delta L}{\delta\dot{x}}) = 0 \quad (29)$$

$$m\ddot{x} - mg = 0 \quad (30)$$

The components of forces in two directions , x and y:

$$y - l\sin(\theta) - r = y_{point} \quad (31)$$

$$\Delta y = (y - \frac{\theta}{|\theta|}\frac{l}{2}\sin(\theta) - r \quad (32)$$

$$\Delta x = x - \frac{\theta}{|\theta|}\frac{l}{2}\cos(\theta) - x_{point} \quad (33)$$

$$m\ddot{y} = (-k\Delta y^{\frac{3}{2}} - \delta(\Delta\dot{y}^{\frac{3}{2}}) \quad (34)$$

$$m\ddot{x} = (-k\Delta x^{\frac{3}{2}} - \delta(\Delta\dot{x}^{\frac{3}{2}}) \quad (35)$$

$$I\ddot{\theta} = -N\frac{\theta}{|\theta|}\frac{l}{2}\cos(\theta) \quad (36)$$

The angular velocity in x, y and Z (Fig. 4):

$$\omega = \frac{\Delta\theta}{\Delta t} \quad (37)$$

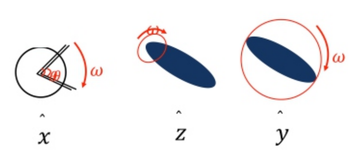


Fig. 4: Angular velocity of the capsule in x, y and z directions

$$I_z = \frac{1}{2}mr^2 + \frac{2}{10}mr^2 \quad (38)$$

$$I_x = I_y = \frac{1}{12}m(3r^2 + h^2) + \frac{2}{10}mr^2$$

$$I_z = \frac{1}{2}mr^2 + \frac{2}{6}mr^2 \quad (39)$$

$$I_x = I_y$$

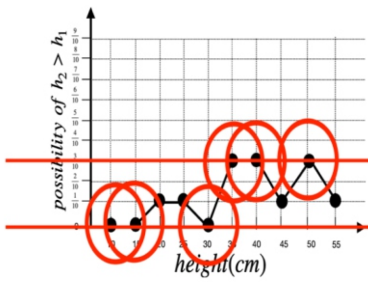
$$Iz = \frac{1}{12}m(3r^2 + h^2) + \frac{2}{6}mr^2 \quad (40)$$

For finding the ideal height several experiments are done in different heights.

4 Results and Conclusions

To write the PDF (Probability Distribution Function), the capsule-shaped object is spined 10 times in different angular velocities in 10 different heights according to the experimental setup (Figs.5,6, 7 and 8).

$h_1 = 10$	$P(x)_1 = 0$	$h_6 = 35$	$P(x)_6 = \frac{3}{10}$
$h_2 = 15$	$P(x)_2 = 0$	$h_7 = 40$	$P(x)_7 = \frac{3}{10}$
$h_3 = 20$	$P(x)_3 = \frac{1}{10}$	$h_8 = 45$	$P(x)_8 = \frac{1}{10}$
$h_4 = 25$	$P(x)_4 = \frac{1}{10}$	$h_9 = 50$	$P(x)_9 = \frac{3}{10}$
$h_5 = 30$	$P(x)_5 = 0$	$h_{10} = 55$	$P(x)_{10} = \frac{1}{10}$

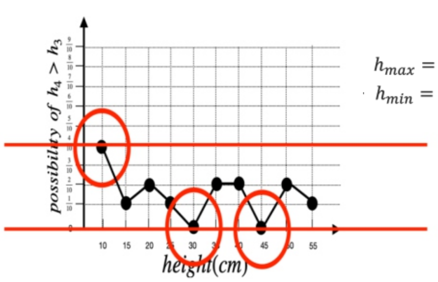


$h_{max} = 35,40,55$
$h_{min} = 10,15,30$

Fig.5: Probability versus height

$p(h_2 > h_1) = 35, 40, 55\ cm$

$h_1 = 10$	$P(x)_1 = \frac{4}{10}$	$h_6 = 35$	$P(x)_6 = \frac{2}{10}$
$h_2 = 15$	$P(x)_2 = \frac{1}{10}$	$h_7 = 40$	$P(x)_7 = \frac{2}{10}$
$h_3 = 20$	$P(x)_3 = \frac{2}{10}$	$h_8 = 45$	$P(x)_8 = 0$
$h_4 = 25$	$P(x)_4 = \frac{1}{10}$	$h_9 = 50$	$P(x)_9 = \frac{2}{10}$
$h_5 = 30$	$P(x)_5 = 0$	$h_{10} = 55$	$P(x)_{10} = \frac{1}{10}$

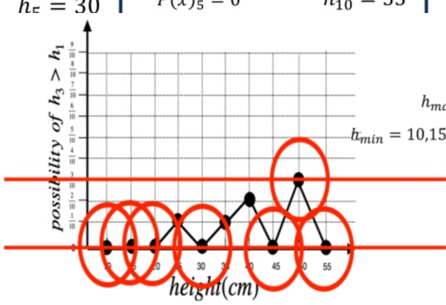


$h_{max} = 25$
$h_{min} = 30,45$

Fig.6: Probability versus height

$p(h_4 > h_3) = 25\ cm$

$h_1 = 10$	$P(x)_1 = 0$	$h_6 = 35$	$P(x)_6 = \frac{1}{10}$
$h_2 = 15$	$P(x)_2 = 0$	$h_7 = 40$	$P(x)_7 = \frac{2}{10}$
$h_3 = 20$	$P(x)_3 = 0$	$h_8 = 45$	$P(x)_8 = 0$
$h_4 = 25$	$P(x)_4 = \frac{1}{10}$	$h_9 = 50$	$P(x)_9 = \frac{3}{10}$
$h_5 = 30$	$P(x)_5 = 0$	$h_{10} = 55$	$P(x)_{10} = 0$



$h_{max} = 25$
$h_{min} = 10,15,20,30,45,55$

Fig.7: Probability versus height

$p(h_3 > h_1) = 25\ cm$

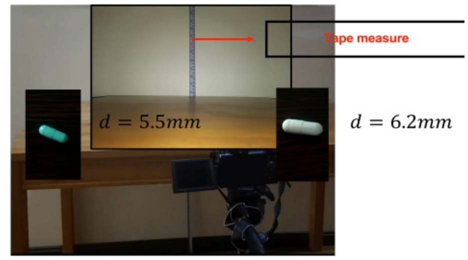


Fig. 8: Experimental Setup

Maximum and minimum of the angular velocities in our experiment are calculated in different collision times as shown in figure (9).

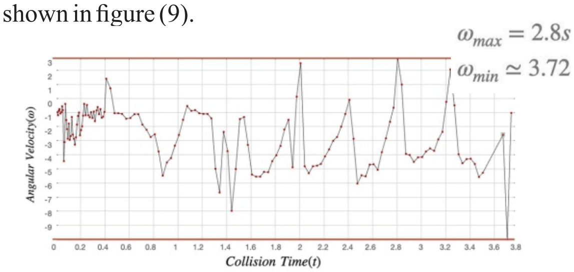


Fig. 9: Angular Velocity versus collision time

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